Student Seminar:
The Sylvester-Gallai Theorem and Its Relatives
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A Survey on the Theorem: Part 1
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Oliver Huggenberger
08-915-514

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1 Sylvester's Problem

The underlying question to the Sylvester-Gallai Theorem has already been posed by Sylvester in 1893: "Let a finite set of points in the plane have the property that the line through any two of them passes through a third point of the set. Must all the points lie on one line?"

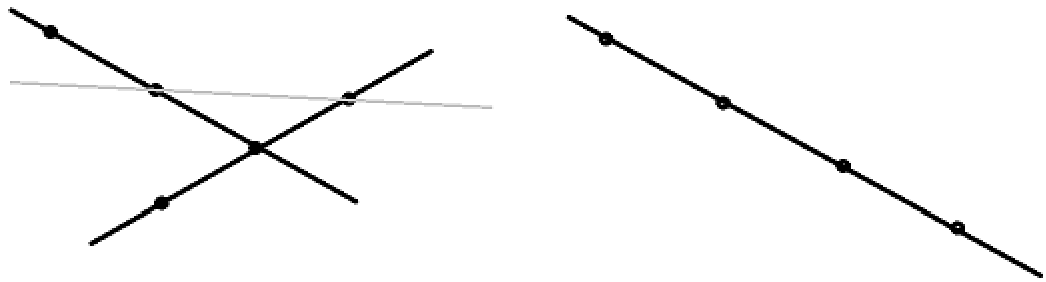

Figure 1: Does a set of noncollinear points always allow a line through exactly two points? This question (posed by Erdös, 1943) is equivalent to the initial one.

The answer is yes, at least for the ordinary real plane, which was proved about half a century later by various mathematicians. Other results may occur for some finite geometries or the complex case.

Definition: Let $P = \{p_1, p_2, ..., p_n\}$, $n \geq 3$ be a set of noncollinear points in the plane. A line that contains two or more points of P is called a *connecting line* and is *ordinary*, if it contains exactly two points.

Theorem 1.1: (Sylvester-Gallai)
Every such set P determines at least one ordinary line.

2 Various Proofs

2.1 Proof by Gallai (1944)

This affine proof was one of the first and gave name to the theorem. It was established by Gallai in 1944. First we introduce the concept of a projective transformation, which is key to the proof.

Let's denote the i-dimensional Euclidean space as E^i then we can define:

Definition: A projective transformation is a mapping $\Theta : E^i \rightarrow E^j$ of the form $\Theta(x) = \frac{Ax+b}{c^T x+d}$, for a linear map A, vectors b, c and a scalar d.

Example: Let $P \subseteq E^2$ be a set like in figure 2. We apply the transformation

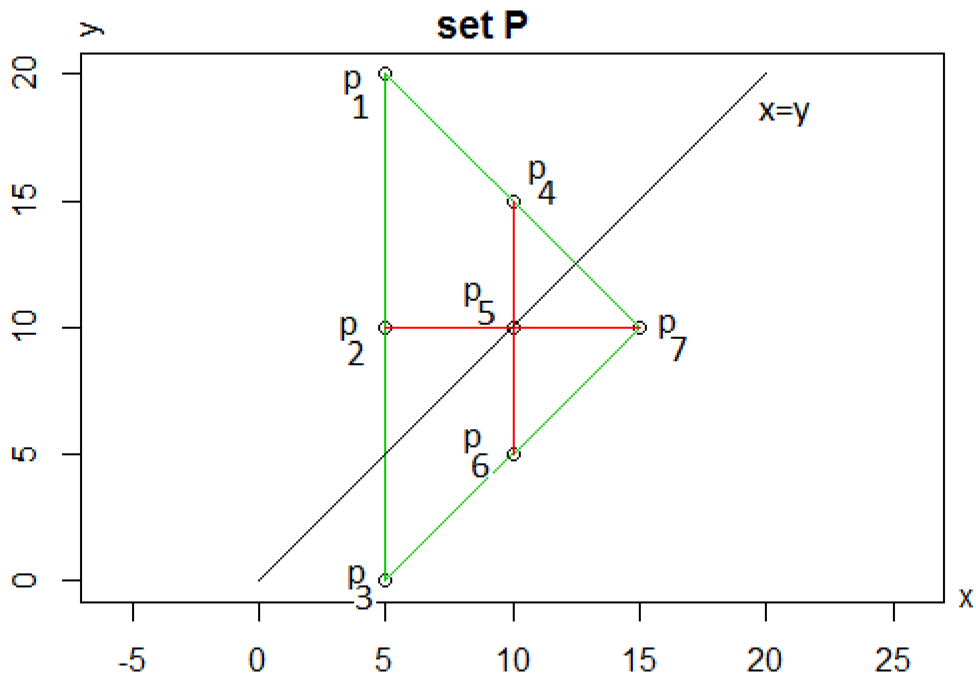

Figure 2: The set P consists of points $p_1, ..., p_7$. The red lines indicate ordinary lines in P through the point p_5. Green lines are ordinary lines not through p_5.

$\Theta_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{x-y} \begin{pmatrix} x+2y+20 \\ 3x+4y-20 \end{pmatrix}$ on every point $p_i \in P$. This corresponds to a projective transformation with $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 20 \\ -20 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $d = 0$. If we transform P by Θ_1, then the whole line $x = y$, including p_5, is projected to infinity. The projective image $\Theta_1(P)$ then looks like in figure 3.

We immediately see that in the projective image the lines containing p_5 are all parallel to $x = y$ and that collinear points in P still lie on a common connecting line.

That collinearity is also preserved for general sets P will be verified hereafter in Corollary 2.1 and Lemma 2.2.

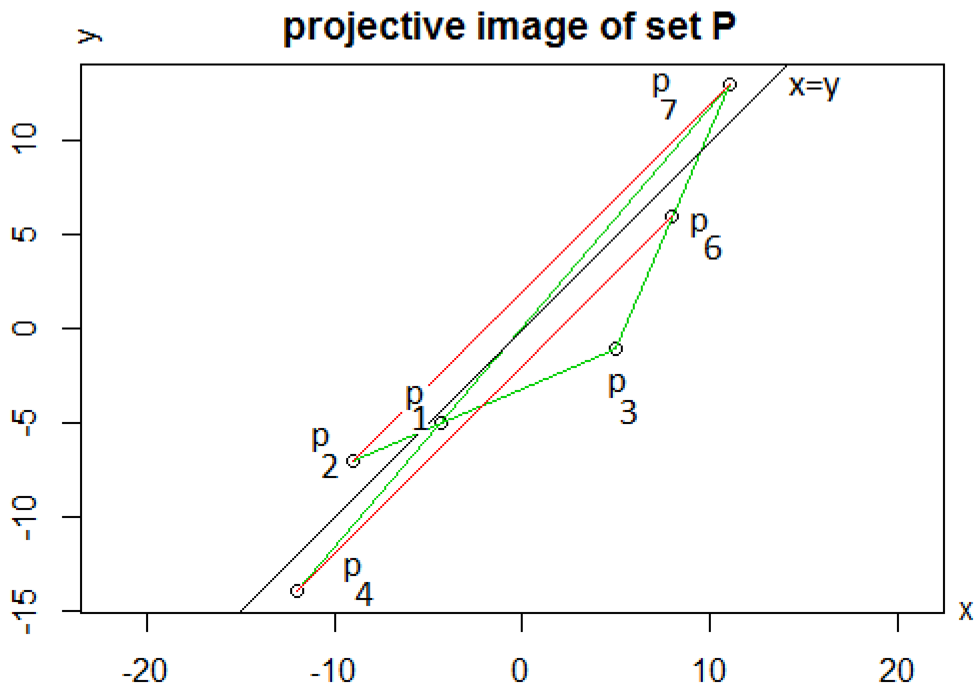

Figure 3: The projective image $\Theta_1(P)$

Definition: A set $\hat{P} \subseteq E^2$ is called affinely dependent, if there exists a combination of $p_1, ..., p_r \in \hat{P}$ and $\lambda_1, ..., \lambda_r \in \mathbb{R}$, such that $\lambda_1 p_1 + ... + \lambda_r p_r = 0$ and $\lambda_1 + ... + \lambda_r = 0$.

Corollary 2.1: A set $\hat{P} = \{p_1, p_2, p_3\}$ is affinely dependent, if and only if it is collinear.
Proof: If and only if $\hat{P}$ is collinear, we can choose $\lambda_1,\ \lambda_2 \in \mathbb{R} - \{0\}$, such that $\lambda_1(p_1 - p_3) + \lambda_2(p_2 - p_3) = 0$. Moreover for $\lambda_3 = -(\lambda_1 + \lambda_2)$ we have:

$$\begin{aligned} 0 &= \lambda_1(p_1 - p_3) + \lambda_2(p_2 - p_3) \\ &= \lambda_1 p_1 + \lambda_2 p_2 - (\lambda_1 + \lambda_2) p_3 \\ &= \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3. \end{aligned} \tag{2.1}$$

Hence $\hat{P}$ fulfills the definition of affine dependence if and only if it is collinear.

Lemma 2.2: If $\hat{P}$ is an affinely dependent set of points in E^2 and $c^T p + d \neq 0$ for all $p \in \hat{P}$, then $\Theta(\hat{P})$ is affinely dependent too.
Proof: See [McMullen], page 19-20.

From Corollary 2.1 and Lemma 2.2 we derive that every three collinear points in P are still collinear after the projective transformation. In other words, Θ maps connecting lines on connecting lines.

We now give Gallai's proof of the Silvester-Gallai theorem:

Proof: (Theorem 1.1)
Choose any point $p_1 \in P$. If it is on an ordinary line we are done. Otherwise we project a line through p_1, which doesn't intersect with other points in P, to infinity. As we have seen before, this leaves collinear points collinear and the lines through p_1 form a set of parallel lines. The following corollary determines an ordinary line and finishes the proof.

Corollary 2.3: The connecting line l not through p_1 that includes the smallest angle a with the set of connecting lines containing p_1 is ordinary.

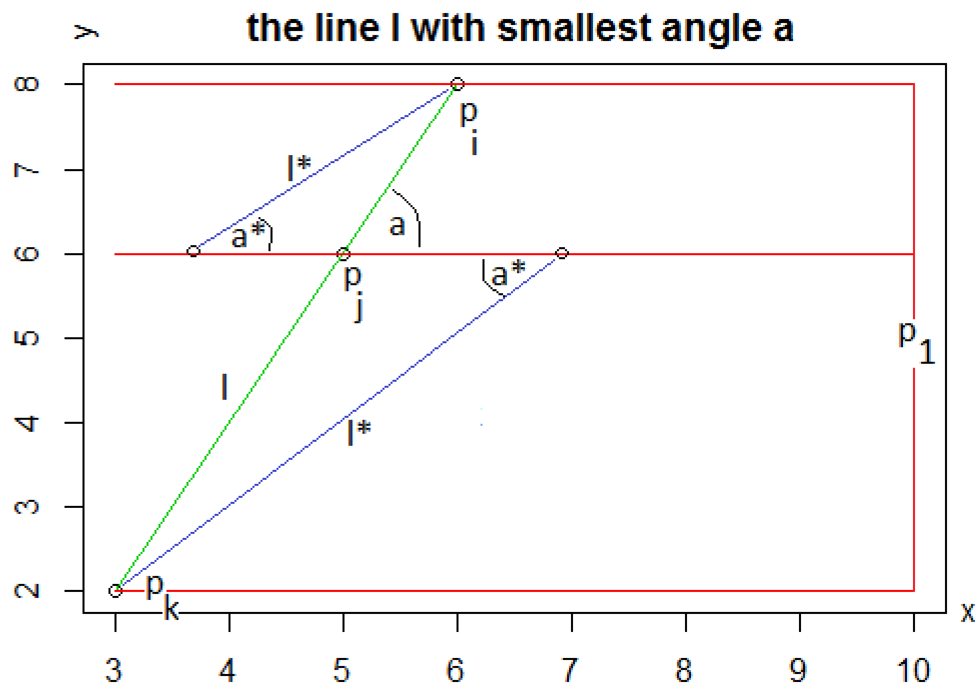

Figure 4: The (green) line l has to be ordinary. Otherwise there existed a third point on the line $\overline{p_j p_1}$, which allows for a line l^* with smaller angle a^*.

Proof: Suppose not, then l has at least three points $p_i, p_j, p_k \in P$ on it. Each of them again lies on a connecting line through p_1 which is not ordinary. Hence we end up in a situation like in figure 4. In that case, we can always construct a line l^* with a smaller angle a^*. With that contradiction we have proved the corollary.

2.2 Proof by Kelly (1948)

This Euclidean proof by Kelly in 1948 is quite simple and illustrative:

Consider again the noncollinear finite set P. Let $S(P)$ be the set of connecting lines in P. For a $p \in P$ and a $s \in S(P)$ not containing p one can determine a perpendicular distance of point p to line s. Denote (s^*, p^*) as the pair with the smallest such distance, then s^* has to be ordinary. Otherwise one finds a smaller distance (see figure 5).

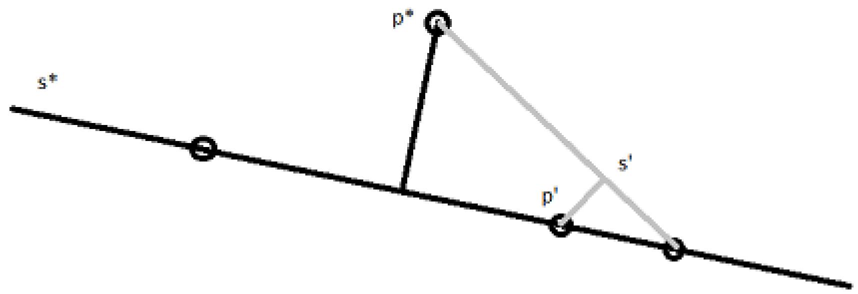

Figure 5: If s^* wouldn't be ordinary, there existed a pair (s', p') with smaller distance.

2.3 Proof by Steinberg (1944)

The projective proof of Steinberg in 1944 was a springboard for later developments:

Let P and $S(P)$ be as before. Choose a $p \in P$, if it is on an ordinary line, we are done. Otherwise choose a line l, such that $l \cap P = \{p\}$. Denote the finite intersection points $l \cap S(P)$ cyclically as $x_1, ..., x_k$, such that there lies no other x_j in the segment $[p, x_1]$. Then the line $s \in S(P)$ that intersects l in point x_1 must be ordinary. Otherwise one could find another point in $[p, x_1]$ (see figure 6).

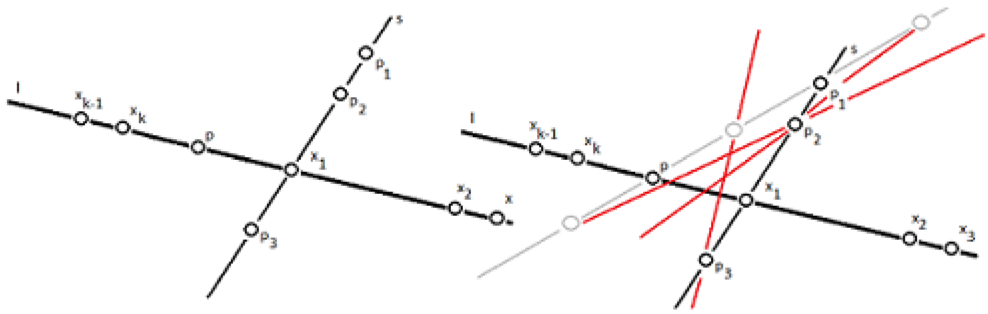

Figure 6: Assume s is not ordinary. Denote the points on s as p_1, p_2 and p_3. If the (grey) line through p_1 and p is not ordinary, then it allows for a configuration where a (red) line intersects l in the segment $[p, x_1]$. This contradiction finishes the proof.

3 Proofs of the Dual Theorem

If we denote the number of ordinary lines determined by P as $m(P)$ and define $m(n) = \min_{|P|=n} m(P)$, Sylvester's theorem states:

$$m(n) \geq 1, \text{for } n \geq 3. \tag{3.1}$$

Melchior (1940) and others even proved a stronger statement:
Theorem 3.1: $m(n) \geq 3, \text{for } n \geq 3$

The duality that is used for the proof is introduced next.

3.1 The Duality

We can translate the definitions and notations from before by using the duality from figure 7:

a) P is a finite set of points not all on one line
b) a connecting line (determined by P) contains two or more points of P
c) $S(P)$ is the set of connecting lines determined by P
d) an i-line, $i \geq 2$, is a connecting line containing exactly i points of P
e) a 2-line is called ordinary
f) $t_i(P)$ denotes the number of i-lines determined by P
g) $|S(P)| = \sum_{i \geq 2} t_i(P)$
h) $m(P) = t_2(P)$
i) $m(n) = \min_{|P|=n} m(P)$

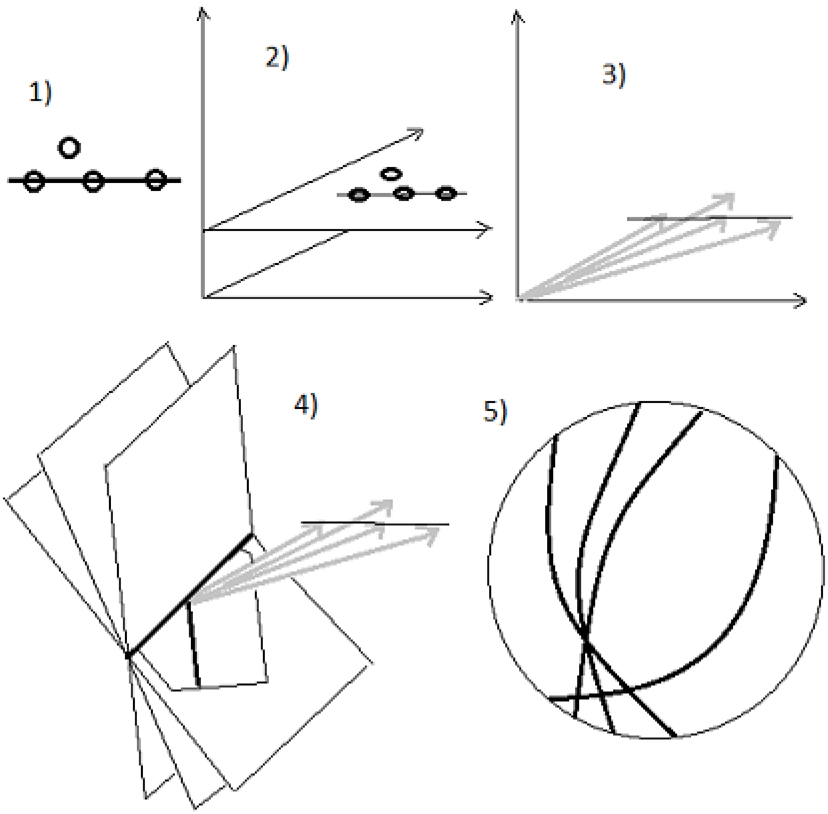

Figure 7: The point configuration in 1) is dual to the one-sphere arrangement in 5). One sees that by constructing vectors $(p_i, 1)$ and cutting the orthogonal hyperplanes with the unit sphere.

In the dual configuration this corresponds to:

a) L is a finite set of lines not all through one point
b) a point of intersection (determined by L) lies on two or more lines of L
c) $V(L)$ is the set of points of intersection determined by L
d) an i-point, $i \geq 2$, is a point lying on exactly i lines of L
e) a 2-point is called simple
f) $v_i(L)$ denotes the number of i-points determined by L
g) $|V(L)| = \sum_{i \geq 2} v_i(L)$
h) $m'(L) = v_2(L)$
i) $m'(n) = \min_{|L|=n} m'(L)$

If L is the dual of P we therefore have:

$$v_i(L) = t_i(P). \tag{3.2}$$

This property is used to derive Melchior's proof of the Sylvester-Gallai theorem.

3.2 Proof by Melchior(1940)

For this proof we consider the set L. We denote the points, edges and faces in the sphere arrangement as $V(L)$, $E(L)$ and $F(L)$. These numbers satisfy the Euler-Poincaré formula for the real projective plane, which is:

$$V(L) - E(L) + F(L) = 1. \tag{3.3}$$

Let $f_i(L)$ be the number of faces having exactly i sides, then:

$$\begin{aligned} V(L) &= \sum_{i \geq 2} v_i(L), \\ F(L) &= \sum_{i \geq 3} f_i(L), \\ 2E(L) &= \sum_{i \geq 3} i f_i(L) = 2 \sum_{i \geq 2} i v_i(L). \end{aligned} \tag{3.4}$$

Example: Let L be a set like in figure 8. Then one can calculate:

$$\begin{aligned} &V(L) = v_2(L) + v_3(L) = 3 + 1 = 4, \\ &F(L) = f_3(L) = 6, \\ &2E(L) = 3f_3(L) = 2[2v_2(L) + 3v_3(L)] = 2(6+3) = 18. \end{aligned} \tag{3.5}$$

And we can verify Euler-Pointcaré: $V(L) - E(L) + F(L) = 4 - 9 + 6 = 1$.

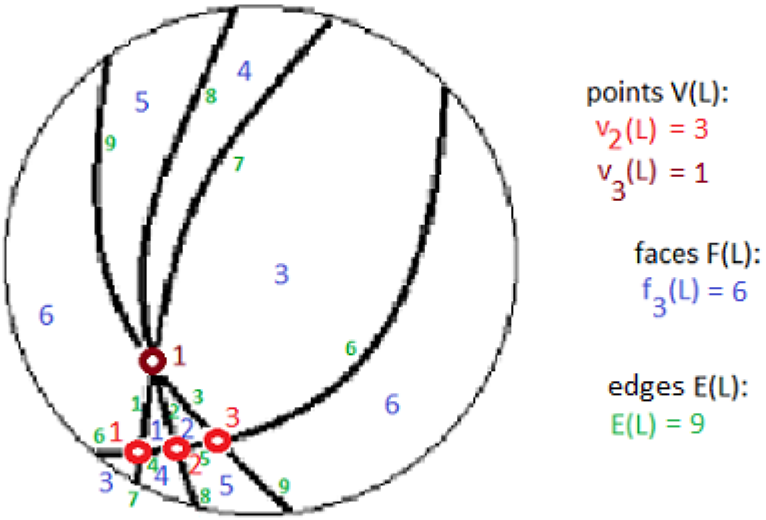

Figure 8: Example of counting points, faces and edges in the real projective plane.

Proof : (Theorem 3.1)
Starting with Euler-Pointcaré we get:

$$
\begin{aligned}
3 &= 3V(L) - E(L) + 3F(L) - 2E(L), \\
&= 3\sum_{i\geq 2} v_i(L) - \sum_{i\geq 2} i v_i(L) + 3\sum_{i\geq 3} f_i(L) - \sum_{i\geq 3} i f_i(L), \\
&= \sum_{i\geq 2}(3-i)v_i(L) + \sum_{i\geq 3}(3-i)f_i(L)\ ,
\end{aligned} \tag{3.6}
$$

which we can rewrite to:

$$v_2(L) \geq 3 + \sum_{i\geq 4}(i-3)v_i(L). \tag{3.7}$$

By duality we get:

$$t_2(P) \geq 3 + \sum_{i\geq 4}(i-3)t_i(P) = 3 + t_4(P) + 2t_5(P) + 3t_6(P) + ... \tag{3.8}$$

Hence, this proves the equation (4.2) which is even stronger than the ordinary Sylvester-Gallai theorem.

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