The Sylvester-Gallai Theorem, the Monochrome Line Theorem and Generalizations

Report for a Seminar on the Sylvester-Gallai Theorem
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In this report, we look at further variations on Sylvester’s original problem. Specifically, we will discuss the results presented in the sections 5 and 6 of the survey paper of P. Borwein and W. O. J. Moser [7]. For completeness we restate the original Sylvester-Gallai Theorem, presented in two equivalent formulations.

**Theorem.** (Sylvester-Gallai, version 1) Let \( P \) be a finite collection of points in the plane. If \( P \) is noncollinear then there exists an ordinary line, i.e. a line which intersects exactly two points of \( P \).

**Theorem.** (Sylvester-Gallai, version 2). Let \( P \) be a finite collection of points in the plane. If \( P \) has the property that for any two distinct points there exists a distinct third point which is collinear with the first two, then the points of \( P \) must lie all on one line.

**Projective plane, dual representation**

We clarify here several concepts which will be used throughout the discussion. In this text the word “plane” is meant to mean either the Euclidean plane (which can be thought of as \( \mathbb{R}^2 \), although the geometry here does not involve coordinates) or the projective plane. Although a formal definition of the projective plane and the laws of projective geometry can be made, we will be satisfied here to either model the projective plane using the Euclidean plane with additional “ideal” points at infinity in every direction, at which all parallel lines in that direction are said to cross, or model the projective plane by taking the unit sphere in \( \mathbb{R}^3 \) and identifying all antipodal points with one another. The first model will be called the *Euclidean projective plane*, the second model will be called the *spherical projective plane*. One can pass from one model to the other by imagining the Euclidean projective plane as “embedded” in \( \mathbb{R}^3 \) as the plane \( F = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 1\} \) together with the ideal points at infinity, one for each direction in the plane \( F \). To each point in \( F \) we can associate the unique line passing through that point and the origin. To each of the ideal points at infinity, we associate the line through the origin which lies in the plane \( \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\} \) and has the direction represented by that ideal point. In this way we have a one to one correspondence between all points of \( F \) and all lines in \( \mathbb{R}^3 \) through the origin, which in turn, stand in one to one correspondence with the antipodal pairs of points of the unit sphere in \( \mathbb{R}^3 \), since each line through the origin defines exactly one such pair by its intersection with the sphere.

When working in higher dimensions, the term “n-space” will be used to denote n-dimensional projective space, which is a direct generalization to higher dimensions of the Euclidean projective plane described above, i.e. we simply associate to each direction in Euclidean space an additional ideal point “at infinity” corresponding to this direction.

One can also use the unit sphere in \( \mathbb{R}^3 \) to model a “dualization” of the projective plane, which we will call the *dual projective sphere*. This representation is
called dual because there is a one to one correspondence between geometric objects and relationships in the projective plane and those on the dual projective sphere. Thus theorems formulated in one setting can be translated to the other, giving one the freedom to choose which setting is best suited for solving a given problem. A formal proof of this dual relationship will not be undertaken, but again we have a hands-on recipe for passing between the projective plane and the dual projective sphere.

To see this we start with the spherical projective plane, where for each pair of antipodal points we have a unique line passing through these points and the origin. For each such line there is then a unique plane passing through the origin, which is perpendicular to that line. Each such plane corresponds to exactly one great circle on the unit sphere, given by the intersection of that plane and the sphere. In this way a one to one correspondence is established between points in the spherical projective plane and great circles on the 3-sphere. Note that geometric information on the dual projective sphere is recorded doubly since the configuration is point symmetric with respect to the origin. For practical purposes it thus suffices to regard only half of the dual projective sphere, which gives a “dual projective hemisphere”. One could give a list of the correspondences between the projective plane and the dual projective sphere; for our purposes we will only need the following: points in the projective plane correspond to great half-circles on the dual projective sphere, and a line connecting two or more points in the projective plane corresponds to the intersections of the great half-circles on the projective sphere corresponding to those points.

Generalizing the Sylvester-Gallai theorem

One direction in which one can generalize Sylvester’s problem is to replace the finite number of points with a finite collection $S = \{S_1, ..., S_n\}$ of sets in the plane. The question then is what conditions on the sets must be made in order to still guarantee the existence of an ordinary line, whereby we now say a line is ordinary if it intersects exactly two members of the collection $S$ (the old definition of ordinary corresponds to the case when each set in $S$ contains only one point). This generalization to sets can also be combined with the generalization to $n \in \mathbb{N}$ dimensions, in which case one can ask about the existence of either ordinary lines or ordinary affine linear subspaces of any dimension $m$, where $1 \leq m \leq n - 1$. For semantic ease, an $m$-dimensional affine linear subspace will be called an $m$-flat.

Sets in the plane

We will first consider generalizations pertaining to a collection of sets $S$ in the plane. As a first step one may consider the case where each set $S_i \in S$ may consist of finitely many distinct points, i.e. the case where $|\bigcup S_i| =: |S|$ is finite. The following family of examples show that this condition is not restrictive enough, at least in two dimensions, i.e one can find arrangements of points
in the plane, grouped into \( n \) sets, such that there are no ordinary lines. The simplest example can be seen in figure 1. This is one of a family of examples which all arise from the following method of construction. Given \( n \in \mathbb{N} \), draw a regular \( 2n \)-gon in the plane and assign each vertex alternately as being part of the set \( S_1 \) or \( S_2 \), such that no two vertexes from the same set are adjacent. Now create a third set of points \( S_3 \) by assigning to each pair of parallel sides of our figure a point at infinity corresponding to the direction of these parallel sides. Thus \( S_3 \) contains \( n \) different points at infinity. An illustration is given in figure 2.1 for the case \( n = 4 \). Every such example in the projective plane can also be adapted such that no points at infinity are involved, as seen in figure 2.2.

These examples show that further restrictions must be made in order to guarantee the existence of an ordinary line. In the following we discuss what is possible when we assume that at least one of the sets of \( S \) is infinite in cardinality, i.e. when \( |S| \) is infinite. The following examples show that this alone is not enough, and inform us as to what further conditions on the sets of \( S \) might be necessary. Firstly, the simple example of three parallel lines (extending infinitely) as in figure 3.1 shows that we run into trouble if the elements of \( S \) can all be unbounded. This example relies though on all elements of \( S \) being unbounded, and it is not discussed in [7] what happens when one requires at least one element of \( S \) to be bounded: for example in figure 3.2 an ordinary line does exist.
Even if one requires all elements of $S$ to be bounded, the following example shows that other problems can arise. Let $S$ consist of three sets $S_1, S_2$ and $S_3$, constructed by choosing first three sets $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3$ each dense in $[0,1]$ on the first coordinate axis of $\mathbb{R}^2$, and then letting $S_i$ be the set $\tilde{S}_i$ together with a copy of $[0,1]$ in the second coordinate direction attached to each point. A drawing would look something like three non-overlapping barcodes covering the unit square $[0,1]^2$. It is clear that any line intersecting to of these sets must intersect a third, due to the density of the $\tilde{S}_i$. This example gives a motivation for make topological restrictions. It can be shown that requiring all elements of $S$ to be open does not suffice; requiring all sets in $S$ to be closed does on the other hand give results. Here is a short chronological list of developments that were made in this direction:

**Proposition.** (Grünbaum 1956 [13]) Assume $|S|$ is infinite and all elements of $S$ are bounded. If each element of $S$ is closed and connected, then there exists an ordinary line.

**Proposition.** (Edelstein 1957 [8]) Assume $|S|$ is infinite and all elements of $S$ are bounded. If each element of $S$ is closed and consists of finitely many connected components, then there exists an ordinary line.

**Proposition.** (Herzog, Kelly 1960 [15]) Assume $|S|$ is infinite and all elements of $S$ are bounded. If each element of $S$ is closed, then there exists an ordinary line.

**Higher Dimensions**

We now assume that $S$ is a finite collection of sets in $n$-space, where $n \geq 1$. As in the 2-dimensional case, we can find an example in 3 dimensions where $S$ spans all of 3-space, yet does not define an ordinary line. A counterexample is given by the so-called desmic configuration, seen in figure 4. In 1966, Edelstein and Kelly [11] conjectured that this figure was the only type of configuration of a finite collection of finite sets spanning 3-space which does not define an ordinary line. This conjecture, known as the “desmic conjecture”, was proven by Borwein in 1983 [4] to be indeed true: any configuration of finitely many finite sets spanning 3-space and containing no ordinary line is projectively equivalent to the desmic configuration.
What conditions on \( S \) do then guarantee an ordinary line? The result of Herzog and Kelly above for dimension 2 turns out to hold in any finite dimension. In their paper [15] in 1960, Herzog and Kelly show that when \( |S| \) is infinite, non-collinear, and each element of \( S \) is compact, then there exists an ordinary line. In 1963, Eldelstein, Herzog and Kelly [10] show that in fact, with the same assumptions, it even follows that there exists a hyperplane (i.e. a flat of dimension one less than the space) which intersects exactly two different sets of \( S \).

Generalizing these results as well as answering the question of whether examples like the desmic configuration exist in dimensions higher than 3, Edelstein and Kelly published the following theorem in 1966 [11]:

**Theorem 1.** If \( S = \{S_1, ..., S_n\} \) is a finite collection of two or more non-empty compact sets in a real normed linear space, then at least one to the following holds:

1. There exists a hyperplane intersecting precisely two of the sets.
2. \( \bigcup S_i \) spans a space of dimensions 1.
3. \( \bigcup S_i \) is finite and spans a space of dimension 2 or 3.

We see that this generalizes the previous result of Edelstein, Herzog and Kelly since assuming \( S \) is infinite and non-collinear corresponds to the situation when 2. and 3. of Theorem 1 do not hold. Alternately, if we assume \( |S| \) is finite, yet spans a space of dimension 4 or higher, we are again in the situation where 2. and 3. of Theorem 1 are not fulfilled, and thus an ordinary hyperplane exists (and therewith an ordinary line as well).

**The monochrome line theorem**

Instead of considering generalizations of the idea of an ordinary line, we know consider the following somewhat different question. Given two disjoint non-empty finite sets \( R \) and \( B \) of points in the plane, does there exist a line which intersects at least two points of one of these sets and none of the other? Such a line we shall call monochromatic, and think of the set \( R \) as a set of red points and \( B \) as a set of blue points. Clearly, for a monochromatic line to exist \( S := R \cup B \) cannot lie all on a line, since otherwise any line would have to intersect.
both \( R \) and \( B \). Also, \(|S| \geq 3\) is also obviously necessary for the existence of a monochromatic line. It turns out that these too basic assumptions are in fact sufficient as well:

**Theorem 2.** *(Monochrome line theorem)* Let \( R, B \) be two non-empty, disjoint, finite sets of points in the plane. If \( S := R \cup B \) contains at least three points and is non-collinear, then a monochromatic line exists.

We will give two different proofs, the first of which is due to Motzkin in 1967, who is said to have been the first to prove this statement.

**Proof 1:** We will solve the dual problem using the dual projective hemisphere. For convenience we shall use the word arc to denote great-half circles. Here is the dual statement of the theorem for this setting: If \( R, B \) are two non-empty, disjoint, finite sets of arcs in the projective hemisphere such that \( S := R \cup B \) contains at least three arcs and such that the arcs in \( S \) are not all concurrent, then there exists a monochromatic vertex. To prove this, we proceed indirectly, assuming there exists no monochromatic vertex and then finding a contradiction. This is for convenience and should not hide the fact that the proof could be formulated constructively and used to give an algorithm which finds a monochrome vertex.

**Claim.** We can find three arcs which are not concurrent and not all of the same color.

**Proof.** Since \( S \) contains at least 3 arcs, we can find two arcs \( A \) and \( B \) of the same color, say blue. If all red arcs passed through \( A \cap B \), then there would exist a blue arc \( Z \) which does not pass through \( A \cap B \), since \( S \) is non-concurrent. Because any two arcs intersect precisely once (by construction of the dual projective hemisphere), this means that the intersection of \( Z \) with \( A \) or \( B \) would not touch any red lines, since these intersect \( A \) and \( B \) in \( A \cap B \). The intersection of \( Z \) with \( A \) or \( B \) would then be monochromatic, but this cannot be by assumption. Thus there must exist a red arc \( C \) which does not pass through \( A \cap B \). The arcs \( A, B \) and \( C \) are then nonconcurrent and bichromatic.

Now, since \( A \) and \( B \) are blue, the vertex \( A \cap B \) must have a red arc \( D \) passing through it.

**Claim.** Without loss of generality, we can assume that the arcs \( A, B \) and \( C \) form a triangular configuration such that \( D \) intersects \( C \) in between \( A \cap C \) and \( B \cap C \), as shown in figure 5.2.

**Proof.** Our representation of the original point sets in the statement of the theorem using the dual hemisphere representation comes from first dualizing to the dual projective sphere and then simplifying this by identifying antipodal points and regarding only one hemisphere. In doing so it does not matter which hemisphere of the dual projective sphere we choose to pick out to use as our representation; in other words it does not matter where we choose to set the “equator”. If one visualizes the dual projective sphere as a globe, then any given choice of hemisphere corresponds to a certain visual line of perspective,
or equivalently, a single fixed perspective and different rotations of the globe. It is visualized in figures 5.1 and 5.2 how by rotation one can change the order of the points $D \cap C, A \cap C,$ and $B \cap C$ to obtain the desired configuration. Since such rotations preserve the geometric information encoded in the underlying dual projective sphere, we can without loss of generality, rotate at will.

![Figure 5.1](image1.png)  ![Figure 5.2](image2.png)

Now, since the arcs $D$ and $C$ are both red and since $D \cap C$ cannot be monochromatic, there must exist a further blue arc $E$ of $S$ which intersects $D$ and $C$ in $D \cap C$. This arc must intersect either $A$ between $A \cap B$ and $A \cap C$ or intersect $B$ between $A \cap B$ and $B \cap C$. In either case we obtain a configuration just as in the claim above and we can apply the same argumentation again and again iteratively. But since $S$ is finite, this process must end in finitely many steps, leaving us with a monochrome vertex, which contradicts our assumption and completes the proof.

We give a second proof, due to Edmonds, Lovasz and Mendel in 1980 [12].

**Proof 2.** Here again the dual projective hemisphere will be used, but the idea now is to use the Euler characteristic formula for polygons (or more generally, graphs) on the projective hemisphere. In this case the formula is $V - E + F = 1$, where as usual $V$ stands for vertices, $E$ for edges and $F$ for faces.

Assume that there exists no monochrome vertex. We will use the following geometric combinatorial data to produce a contradiction:

- $r_i :=$ number of $i$-gons
- $c :=$ number of corners bounded by two different colors

Observe the following relations:

1. $F = \sum_{i \geq 3} r_i$ since no 2-gons are possible.
2. $2E = \sum_{i \geq 3} ir_i$ since every edge borders exactly two faces.
3. $c \leq 2r_3 + \sum_{i \geq 4} ir_i$ since we observe that 3-gons can have at most 2 bichromatic corners.
4. \( c \geq 4V \) since by assumption every vertex is bichromatic and any two intersecting arcs of different colors produce at least 4 bichromatic corners.

Combining the first 3 relations we have

\[
V = 1 - F + E = 1 - \sum_{i \geq 3} r_i + \frac{1}{2} \sum_{i \geq 3} i r_i = 1 + \sum_{i \geq 3} \left( \frac{i}{2} - 1 \right) r_i
\]

\[
\Rightarrow 4V = 4 + \sum_{i \geq 3} 4\left( \frac{i}{2} - 1 \right) r_i = 4 + \sum_{i \geq 3} (2i - 1) r_i
\]

\[
= 4 + 2r_3 + 4r_4 + 6r_5 + 8r_6 + ...
\]

\[
> 0 + 2r_3 + 4r_4 + 5r_5 + 6r_6 + ...
\]

\[
\geq c
\]

so we find \( c < 4V \), which contradicts relation 4, completing the proof.

We have seen that Sylvester’s question about ordinary lines and the above question concerning monochrome lines could both be answered in the positive. The following then seems natural:

**Question:** Assume \( S = R \cup B \) is a finite non-collinear set of points in the plane, where each point of \( S \) is colored either red or blue. Does there always exist a monochromatic ordinary line?

**Answer:** No. The figure 6 below illustrates this.

![Figure 6](image)

One can though combine the Sylvester-Gallai theorem and the monochrome line theorem into a single statement, as stated in [17], which includes both as a special case:

**Theorem 3.** Let \( S \) be a finite set of points in the plane, each point colored red or blue or both. Assume that for any two points \( A, B \) sharing a color there exists a third collinear point \( C \) of the other color. Then \( S \) is collinear.

Here we have that \( S = R \cup B \), where \( R \) and \( B \) are no longer required to be disjoint. The Sylvester-Gallai theorem corresponds to the case when \( R \cap B = \emptyset \) and the monochrome line theorem follows when \( R = B \).
Generalizations of the monochrome line theorem

We summarize quickly some further results inspired by the monochrome line problem. The first result generalizes from 2 colors and 2 dimensions to n colors and n dimensions:

**Proposition.** (Shannon 1974 [16]) Given n finite, disjoint sets, the union of which spans $\mathbb{R}^n$, then there always exists a monochrome line.

**Proposition.** (Borwein 1982 [3]) If $R, B$ are two disjoint finite sets which span $\mathbb{R}^n$, then one of the following holds:

i) there exists a monochrome line spanned by $R$

ii) there exists a monochrome hyperplane spanned by $B$

**Remark.** Shannon’s proposition follows from Borwein’s: if $S_1, ..., S_n$ are the disjoint sets in Shannon’s theorem, set $R = S_1$ and $B = S_2 \cup ... \cup S_n$. If i) holds then we are done. If ii) holds then apply Borwein’s theorem again for $R = S_2$, $B = S_3 \cup ... \cup S_n$ and so on.

**Remark.** We cannot expect only ii) to hold in general. For example in $\mathbb{R}^3$ we can find sets $R$ and $B$ which do not define a monochrome plane, as seen in figure 7.

![Figure 7](image_url)

We conclude with a conjecture, which is a generalization of Borwein’s proposition above.

**Conjecture.** (Borwein, Edelstein 1983 [6]) Given two disjoint finite sets $R$ and $B$ whose union spans $\mathbb{R}^{n+m}$, then one of the following holds:

i) there exists an $R$-monochrome $n$-flat

ii) there exists a $B$-monochrome $m$-flat
References